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NON-MEAGER FREE SETS FOR MEAGER RELATIONS ON POLISH SPACES

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Abstract. We prove that for each meager relation $E \subset X \times X$ on a Polish space X there is a nowhere meager subspace $F \subset X$ which is E-free in the sense that $(x, y) \notin E$ for any distinct points $x, y \in F$.

1. Introduction

This paper is devoted to the problem of finding non-meager free subsets for meager relations on Polish spaces. For a relation $E \subset X \times X$, a subset $F \subset X$ is called *E-free* if $(x, y) \notin E$ for any distinct points $x, y \in F$. This is equivalent to saying that $F^2 \cap E \subset \Delta_X$ where $\Delta_X = \{(x, y) \in X^2 : x = y\}$ is the diagonal of X^2.

The problem of finding "large" free sets for certain "small" relations was considered by many authors, see [10], [11], [9], [6], [7]. Observe that the classical Mycielski-Kuratowski Theorem [8, 18.1] implies that for each meager relation $E \subset X^2$ on a perfect Polish space X there is an E-free perfect subset $F \subset X$. We recall that a subset of a Polish space is *perfect* if it is closed and has no isolated points. Nonetheless the following result seems to be new.

Theorem 1. *For each meager relation $E \subset X^2$ on a Polish space X there is an E-free nowhere meager subspace $B \subset X$. Moreover, if the set of isolated points is not dense in X then B may be chosen of any cardinality $\kappa \in [\mathrm{cof}(\mathcal{M}), \mathfrak{c}]$.*

Let us recall that a subspace A of a topological space X

- is *meager in X*, if A can be written as a countable union $A = \bigcup_{n\in\omega} A_n$ of nowhere dense subsets of X;
- is *nowhere meager in X*, if for any non-empty open set $U \subset X$ the intersection $U \cap A$ is not meager in X.

It is clear that a subset $A \subset X$ of a Polish space X is nowhere meager if and only if A is dense in X and contains no open meager subspace. By definition, $\mathrm{cof}(\mathcal{M})$ is the minimal cardinality of a collection $\mathcal{X}$ of meager subsets of the Baire space ω^ω such that for every meager $A \subset \omega^\omega$ there exists $X \in \mathcal{X}$ containing A. It is known [5] that $\mathrm{cof}(\mathcal{M}) = \mathfrak{c}$ under Martin's Axiom, and $\mathrm{cof}(\mathcal{M}) < \mathfrak{c}$ in some models of ZFC, see [4].

Theorem 1 will be proved in Section 3. One of its applications is the existence of a first-countable uniform Eberlein compact space which is not supercompact (see [1, 5.2]), which was our initial motivation for considering free non-meager sets for meager relations. The following simple example shows that the nowhere meager set F in Theorem 1 cannot have the Baire property. We recall that a subset A of a topological space X has *the Baire property* in X if for some open set $U \subset X$ the symmetric difference $A \triangle U = (A \setminus U) \cup (U \setminus A)$ is meager in X.

Example 2. For the nowhere dense relation

$$E = \bigcup_{n\in\omega} \{(x, y) \in \mathbb{R}^2 : |x - y| = 2^{-n}\} \subset \mathbb{R} \times \mathbb{R}$$

on the real line $\mathbb{R}$, each E-free subset $F \subset \mathbb{R}$ with the Baire property is meager.

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Proof. Assuming that F is not meager, and using the Baire property of F, find a non-empty open subset $U \subset \mathbb{R}$ such that $U \triangle F$ is meager and hence lies in some meager F_σ-set $M \subset \mathbb{R}$. Then $G = U \setminus M \subset F$ is a dense G_δ-set in U. By the Steinhaus-Pettis Theorem [8, 9.9], the difference $G - G = \{x - y : x, y \in G\}$ is a neighborhood of zero in $\mathbb{R}$ and hence $2^{-n} \in G - G$ for some $n \in \omega$. Then any points $x, y \in G \subset F$ with $|x - y| = 2^{-n}$ witness that the set $F \ni x, y$ is not E-free. □

Remark 3. By a classical result of Solovay [12], there are models of ZF in which all subsets of the real line have the Baire property. In such models each E-free subset for the relation $E = \bigcup_{n\in\omega}\{(x, y) \in \mathbb{R}^2 : |x - y| = 2^{-n}\}$ is meager. This means that the proof of Theorem 1 must essentially use the Axiom of Choice.

2. Some auxiliary results

We recall [2] that a family $\mathcal{F}$ of infinite subsets of a countable set X is called a *semifilter*, if $A \in \mathcal{F}$ provided $F \subset^* A \subset X$ for some set $F \in \mathcal{F}$. Here $F \subset^* A$ means that $F \setminus A$ is finite. Each semifilter on X is contained in the semifilter $[X]^\omega$ of all infinite subsets of X. The semifilter $[X]^\omega$ is a subset of the power set $\mathcal{P}(X)$ which can be identified with the Tychonoff product 2^X via characteristic functions. So, we can speak about topological properties of semifilters as subspaces of the compact Hausdorff space $\mathcal{P}(X)$. According to Talagrand's characterization of meager semifilters on ω, a semifilter $\mathcal{F}$ on a countable set X is meager (as a subset of $\mathcal{P}(X)$) if and only if $\mathcal{F}$ can be enlarged to a σ-compact semifilter $\tilde{\mathcal{F}} \subset [X]^\omega$. This characterization implies the following:

Corollary 4. *For any finite-to-one map $\phi : X \to Y$ between countable sets, a semifilter $\mathcal{F} \subset \mathcal{P}(X)$ is meager if and only if the semifilter $\phi[\mathcal{F}] = \{E \subset Y : \phi^{-1}(E) \in \mathcal{F}\} \subset \mathcal{P}(Y)$ is meager.*

We recall that a map $f : X \to Y$ between two sets is called *finite-to-one* if for each $y \in Y$ the preimage $\psi^{-1}(y)$ is finite and non-empty. In particular, each monotone surjection $\psi : \omega \to \omega$ is finite-to-one.

A key ingredient of the proof of Theorem 1 in the following proposition.

Proposition 5. *For any meager relation $E \subset 2^\omega \times 2^\omega$ on the Cantor cube 2^ω there is a family $(G_\alpha)_{\alpha<\mathfrak{c}}$ of nowhere meager subsets in 2^ω such that $(G_\alpha \times G_\beta) \cap E = \emptyset$ for any distinct ordinals $\alpha, \beta < \mathfrak{c}$.*

Proof. Using the fact that the points of the Cantor cube 2^ω can be identified with the branches of the binary tree $2^{<\omega} = \bigcup_{n\in\omega} 2^n$, we can find a closed subset $\{A_\alpha\}_{\alpha<\mathfrak{c}}$ of $\mathcal{P}(\omega) = 2^\omega$ which consists of infinite subsets of ω and is almost disjoint in the sense that $A_\alpha \cap A_\beta$ is finite for any distinct ordinals $\alpha, \beta < \mathfrak{c}$. The compactness of $\{A_\alpha\}_{\alpha<\mathfrak{c}}$ in 2^ω implies the existence of a monotone surjection $\varphi : \omega \to \omega$ such that $\varphi(A_\alpha) = \omega$ for all $\alpha < \mathfrak{c}$.

Fix any free ultrafilter $\mathcal{U}$ on ω and for every $\alpha < \mathfrak{c}$ choose an ultrafilter $\mathcal{U}_\alpha$ on ω extending the family $\{A_\alpha \cap \varphi^{-1}[U] : U \in \mathcal{U}\}$. The almost disjoint property of the family $\{A_\alpha\}_{\alpha<\mathfrak{c}}$ guarantees that $\omega \setminus A_\alpha \in \mathcal{U}_\xi$ for any distinct ordinals $\alpha, \xi < \mathfrak{c}$.

Lemma 6. *For every $\alpha < \mathfrak{c}$, the filter*

$$\mathcal{F}_\alpha = \mathcal{P}(\omega \setminus A_\alpha) \cap \bigcap_{\alpha \neq \xi < \mathfrak{c}} \mathcal{U}_\xi$$

is non-meager in $\mathcal{P}(\omega \setminus A_\alpha)$.

Proof. By Corollary 4, the filter $\mathcal{F}_\alpha$ is not meager in $\mathcal{P}(\omega \setminus A_\alpha)$ as its image $\varphi[\mathcal{F}_\alpha] = \{E \subset \omega : \varphi^{-1}[E] \in \mathcal{F}_\alpha\}$ coincides with the ultrafilter $\mathcal{U}$ and hence is not meager in $\mathcal{P}(\omega)$. □

Let $E \subset 2^\omega \times 2^\omega$ be a meager relation on 2^ω. By [3, Theorem 2.2.4], there exist a monotone surjection $\phi : \omega \to \omega$ and functions $f_0, f_1 : \omega \to 2$ such that

$$E \subset \big\{(g, g') \in 2^\omega \times 2^\omega \ : \ \forall^\infty n \in \omega \ \big(g \restriction \phi^{-1}(n) \neq f_0 \restriction \phi^{-1}(n)\big) \ \vee \ \big(g' \restriction \phi^{-1}(n) \neq f_1 \restriction \phi^{-1}(n)\big)\big\}.$$

For every ordinal $\alpha < \mathfrak{c}$ consider the subset

$$G_\alpha = \big\{g \in 2^\omega \ : \exists X_0, X_1 \in \mathcal{U}_\alpha \setminus \bigcup_{\alpha \neq \xi < \mathfrak{c}} \mathcal{U}_\xi \\ \big(X_0 \subset X_1\big) \wedge \big(g \restriction \phi^{-1}[X_0] = f_0 \restriction \phi^{-1}[X_0]\big) \wedge \big(g \restriction \phi^{-1}[\omega \setminus X_1] = f_1 \restriction \phi^{-1}[\omega \setminus X_1]\big)\big\}$$

in the Cantor cube 2^ω.

Lemma 7. *For every ordinal $\alpha < \mathfrak{c}$ the set G_α is nowhere meager in 2^ω.*

Proof. Since G_α is closed under finite modifications of its elements, it is enough to show that G_α is non-meager in 2^ω. Observe that G_α contains the set

$$G'_\alpha = \big\{g \in 2^\omega \colon \exists Y_0 \in \mathcal{U}_\alpha \cap \mathcal{P}(A_\alpha) \ \ \exists Y_1 \in \mathcal{P}(\omega \setminus A_\alpha) \setminus \bigcup_{\alpha \neq \xi < \mathfrak{c}} \mathcal{U}_\xi \\ \big(g \restriction \phi^{-1}[Y_0] = f_0 \restriction \phi^{-1}[Y_0]\big) \wedge \big(g \restriction \phi^{-1}[\omega \setminus (A_\alpha \cup Y_1)] = f_1 \restriction \phi^{-1}[\omega \setminus (A_\alpha \cup Y_1)]\big)\big\}.$$

Indeed, if $g \in G'_\alpha$ is witnessed by Y_0, Y_1, then $X_0 = Y_0$ and $X_1 = A_\alpha \cup Y_1$ are witnessing that $g \in G_\alpha$. Now G'_α may be written as the product $R_\alpha \times H_\alpha$, where

$$R_\alpha = \big\{g \in 2^{\phi^{-1}[A_\alpha]} \ : \exists Y_0 \in \mathcal{U}_\alpha \cap \mathcal{P}(A_\alpha) \ \ \big(g \restriction \phi^{-1}[Y_0] = f_0 \restriction \phi^{-1}[Y_0]\big)\big\}$$

and

$$H_\alpha = \big\{g \in 2^{\phi^{-1}[\omega \setminus A_\alpha]} \ : \exists Y_1 \in \mathcal{P}(\omega \setminus A_\alpha) \setminus \bigcup_{\alpha \neq \xi < \mathfrak{c}} \mathcal{U}_\xi \\ \big(g \restriction \phi^{-1}[\omega \setminus (A_\alpha \cup Y_1)] = f_1 \restriction \phi^{-1}[\omega \setminus (A_\alpha \cup Y_1)]\big)\big\}.$$

Thus it suffices to show that both R_α and H_α are non-meager. By the homogeneity of 2^ω there is no loss of generality to assume that $f_0 \restriction \phi^{-1}[A_\alpha] \equiv 1$ and $f_1 \restriction \phi^{-1}[\omega \setminus A_\alpha] \equiv 1$.

With f_1 as above we see that H_α is simply the set of characteristic functions of elements of the semifilter

$$\mathcal{H}_\alpha = \big\{Z \subset \phi^{-1}[\omega \setminus A_\alpha] \ : \ \exists Y_1 \in \mathcal{P}(\omega \setminus A_\alpha) \setminus \bigcup_{\alpha \neq \xi < \mathfrak{c}} \mathcal{U}_\xi \quad \big(\phi^{-1}[\omega \setminus (A_\alpha \cup Y_1)] \subset Z\big)\big\}$$

on $\phi^{-1}[\omega \setminus A_\alpha]$. Therefore

$$\phi[\mathcal{H}_\alpha] = \big\{T \subset \omega \setminus A_\alpha \ : \ \exists Y_1 \in \mathcal{P}(\omega \setminus A_\alpha) \setminus \bigcup_{\alpha \neq \xi < \mathfrak{c}} \mathcal{U}_\xi \quad \big(\omega \setminus (A_\alpha \cup Y_1) \subset T\big)\big\}.$$

Observe that $Y_1 \in \mathcal{P}(\omega \setminus A_\alpha) \setminus \bigcup_{\alpha \neq \xi < \mathfrak{c}} \mathcal{U}_\xi$ iff $\omega \setminus (A_\alpha \cup Y_1) \in \bigcap_{\alpha \neq \xi < \mathfrak{c}} \mathcal{U}_\xi$, and hence $\phi[\mathcal{H}_\alpha]$ is equal to the filter $\mathcal{P}(\omega \setminus A_\alpha) \cap \bigcap_{\alpha \neq \xi < \mathfrak{c}} \mathcal{U}_\xi$ which is non-meager in $\mathcal{P}(\omega \setminus A_\alpha)$ by Lemma 6, and consequently the filter $\mathcal{H}_\alpha$ is non-meager in $\mathcal{P}(\phi^{-1}[\omega \setminus A_\alpha])$ by Corollary 4. In other words, H_α is a non-meager subset of $2^{\phi^{-1}[\omega \setminus A_\alpha]}$.

The proof of the fact that R_α is non-meager is analogous. However, we present it for the sake of completeness. With f_0 as above we see that R_α is simply the set of characteristic functions of elements of the semifilter

$$\mathcal{R}_\alpha = \{Z \subset \phi^{-1}[A_\alpha] \ : \ \exists Y_0 \in \mathcal{P}(A_\alpha) \cap \mathcal{U}_\alpha \ \big(\phi^{-1}[Y_0] \subset Z\big)\}$$

on $\phi^{-1}[A_\alpha]$. It follows that

$$\phi[\mathcal{R}_\alpha] = \{T \subset A_\alpha \ : \ \exists Y_0 \in \mathcal{P}(A_\alpha) \cap \mathcal{U}_\alpha \ \big(Y_0 \subset T\big)\} = \mathcal{P}(A_\alpha) \cap \mathcal{U}_\alpha$$

is a non-meager ultrafilter on A_α and then $\mathcal{R}_\alpha$ is a non-meager semifilter on $\phi^{-1}[A_\alpha]$ according to Corollary 4. Consequently, R_α is a non-meager subset of $2^{\phi^{-1}[A_\alpha]}$. □

Lemma 8. *For any distinct ordinals $\alpha, \beta < \mathfrak{c}$ we get $(G_\alpha \times G_\beta) \cap E = \emptyset$.*

Proof. Assume conversely that $(G_\alpha \times G_\beta) \cap E$ contains some pair (g_α, g_β). Fix sets X_0^α, X_1^α and X_0^β, X_1^β witnessing that $g_\alpha \in G_\alpha$ and $g_\beta \in G_\beta$, respectively. The intersection $X_0^\alpha \cap (\omega \setminus X_1^\beta)$ is infinite: otherwise $X_0^\alpha \subset^* X_1^\beta$ and $X_1^\beta \in \mathcal{U}_\alpha$, which contradicts the definition of G_β. Thus the set $X_0^\alpha \setminus X_1^\beta$ is infinite and for every $n \in X_0^\alpha \setminus X_1^\beta$ we get $g_\alpha \restriction \phi^{-1}(n) = f_0 \restriction \phi^{-1}(n)$ and $g_\beta \restriction \phi^{-1}(n) = f_1 \restriction \phi^{-1}(n)$, which implies $(g_\alpha, g_\beta) \notin E$. □

This completes the proof of Proposition 5. □

Using the well-known fact that each perfect Polish space X contains a dense G_δ-subset homeomorphic to the space of irrationals ω^ω, we can generalize Proposition 5 as follows.

Proposition 9. *For any meager relation $E \subset X \times X$ on a perfect Polish space X there is a family $(G_\alpha)_{\alpha<\mathfrak{c}}$ of nowhere meager subsets in X such that $(G_\alpha \times G_\beta) \cap E = \emptyset$ for any distinct ordinals $\alpha, \beta < \mathfrak{c}$.*

3. Proof of Theorem 1

Let $E \subset X \times X$ be a meager relation on a Polish space X. If the set D of isolated points is dense in X, then $B = D$ is a required nowhere meager E-free subset of X. So, we assume that the set D is not dense in X. Then the open subspace $Y = X \setminus \bar{D}$ of X is not empty and has no isolated points. Let $\kappa \in [\mathrm{cof}(\mathcal{M}), \mathfrak{c}]$ be any cardinal. By Proposition 9, there is a family $(G_\alpha)_{\alpha<\kappa}$ of nowhere meager subsets in Y such that $(G_\alpha \times G_\beta) \cap E = \emptyset$ for any distinct ordinals $\alpha, \beta < \kappa$.

Let $\mathcal{U}$ be a countable base of the topology of Y and $\mathcal{X}$ be a cofinal with respect to inclusion family of meager subsets in Y of size κ. It is clear that the set $\mathcal{U} \times \mathcal{X}$ has cardinality κ and hence can be enumerated as $\mathcal{U} \times \mathcal{X} = \{(U_\alpha, X_\alpha) : \alpha < \kappa\}$. Since the set D is at most countable and E is meager in $X \times X$, the set $E_0 = \{y \in Y : \exists x \in D \ \ (x, y) \in E \text{ or } (y, x) \in E\}$ is meager in Y. For every ordinal $\alpha < \kappa$ the set G_α is nowhere meager in Y, which allows us to find a point $y_\alpha \in U_\alpha \cap G_\alpha \setminus (X_\alpha \cup E_0)$. Then $B = D \cup \{y_\alpha\}_{\alpha<\kappa}$ is a nowhere meager E-free set in X.

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